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Advertising Differential Games: a Review on Models

Mostafa Jafari[a*], Mohammad Mohammadpour omran[b] and Ehsan Jahani[c]

[a] *Associate Professor, Department of Industrial Engineering , Iran University of Science & Technology, Tehran, Iran.*
[b] *Assistant Professor, Department of Industrial Engineering , Iran University of Science & Technology, Tehran, Iran.*
[c] *PhD student, Department of Industrial Engineering , Iran University of Science & Technology, Tehran, Iran.*

***Keywords*:**

1 Advertising competition
2 Differential game
3 Dynamic game

A B S T R A C T

During the recent years, various research studies have been conducted on modeling the dynamic advertising process in the business realm. The competitive environment of this process has mostly been modeled using game theory, especially differential games with a capability of modeling the conflict in a dynamic environment. The present study aims to survey various types of differential game modeling and present the research gaps in this domain. To this end, 113 papers published between 1960 and 2019 are reviewed. The papers are classified based on five factors: the equilibrium type (Nash/Stackelberg), solution approach (open-loop/closed-loop/feedback), certainty state (deterministic /stochastic), the number of players (duopoly/oligopoly), and time horizon (finite/infinite) and are described using pivot charts. These five factors play a pivotal role in solving dynamic differential games. Finally, the key papers and the available research gaps are recognized and recommendations for performing future studies are presented.

1. Introduction

Advertising is assumed as one of the most important marketing management tools and a conventional communication form to motivate customers to make decisions on purchasing a product or service. A company can utilize traditional mass media such as newspapers, magazines, television advertising, radio advertising, outdoor advertising or direct mail, or via emerging media including websites, text messages, or social networks. With the development of commercial competitions, advertising has proven its role more than ever in the strategic plans of organizations and has become an integral part of marketing programs today. Thus, advertising measurements have emerged in a competitive environment as well. Companies are attempting to advertise their product or service to consumers through advertising media. Nonetheless, budget limitations prevent companies from infinite advertisement [1]. Therefore, a company needs to adopt the best decision for advertising efforts to increase sales and profits of the fiscal year. The impact of advertising on sales, market share, goodwill, etc. depends on time and in other words, it changes over time. Consequently, a company is advised to select its

* Corresponding author, Email: jafari@iust.ac.ir

advertising strategy, which includes deciding on how to make the advertising effort (quantity and time), in a competitive and dynamic environment. Hence, there is a need for a dynamic competitive advertising model to adopt the best decision on advertising strategy. Concerning the aforementioned points, a good number of researchers have modeled dynamic advertising competition using differential dynamic game models. The advantage of using a differential game approach for advertising competition is the evaluation of the available strategies based on the game theory and the use of continuous-time dynamics. Employing the game theory in differential games allows players to predict their rivals' strategies and the dynamic nature of the game theory enables the players to identify not only current revenues but also their future advertising decisions [2]. During the previous years, several review studies have been carried out in the area of dynamic advertising models. Table 1 presents the details of the research regarding the type of the categorization in the modeling approaches.

Table 1. Review papers concerning dynamic advertising models

	Research Objective	Type of Categorization
[3]	A survey on the applications of optimal dynamic control theory in advertising	• Advertising capital models (Nerlov-Arrow) • Sales advertising response models (Vidale-Wolfe) • Micro-models (Diffusion)
[4]	A survey on the application of optimal dynamic control theory in advertising	• The capital created by advertising, price, and quality. • Sales advertising response models • Models for market growth or cumulative sales • Models with more than one state variable in the advertising process • Competitive models
[5]	A survey on differential game models in advertising competitions	• Base model (Vidale-Wolfe, Excess Advertising, Lanchester, Diffusion) • Number of competitors • Solution (Open-loop, Closed-loop, Feedback) • Analysis method (Numerical, Analytical)
[6]	A survey on differential games in marketing	• Advertising models (Market share models, Sales response models, Diffusion models, Advertising goodwill models) • Equilibrium (Nash/Stackelberg), • Solution (Open-loop, Closed-loop, Feedback) • Cooperation (cooperative, non-cooperative) • This study analyzes the pricing models and marketing channel models as well.
[7]	A survey on Stackelberg differential games in advertising channels	• Dynamic (Nerlov-Arrow, Lanchester) • Leader and follower decision variable (advertising effort, price) • Solution (Open-loop, Closed-loop, Feedback)
[1]	A survey on the recent research on dynamic advertising	• Dynamic (Nerlov-Arrow, Lanchester) • Equilibrium (Nash/Stackelberg) • Solution (Open-loop, Closed-loop, Feedback) • Analysis method (Numerical, Analytical) • Time horizon (finite, infinite)
[8]	A survey on models based on the game theory of cooperative advertising	• Decision variable (advertising cost, customer price, quality) • Number and type of market players (manufacturer and retailer)

The current study focuses on the research conducted on dynamic advertising differential games. In other words, the investigated research consists of papers with models based on differential dynamic game models in either duopoly or oligopoly settings. In the general form of such problem, the state variable $x_i(t)$, which is mostly the market share, the market share, or the goodwill of the ith competitor, grows as a differential equation:

$$\dot{x}_i(t) = \frac{dx}{dt} = f\big(x_1, \dots, x_N(t), u_1, \dots, u_N(t)\big) \quad i = 2, \dots, N \tag{1}$$

Where $u_i(t)$ is the decision or control variable of the ith competitor and generally consists of advertising efforts. Also, $f\big(x_1, \dots, x_N(t), u_1, \dots, u_N(t)\big)$ is a function of the state variable and control variable of the player itself or other players that determines how the state variable grows, and N denotes the number of participating players. Under this constraint, each player tends to maximize its performance index, which is typically represented as profit:

$$J_i = \int_0^\infty g\big(x_1, \dots, x_N(t), u_1, \dots, u_N(t)\big)\, dt \quad i = 2, \dots, N \tag{2}$$

By extracting journal papers, conference papers, and work papers of academic institutions in English, a total of 113 papers proposed in the area of differential dynamic games modeling, most of which published in journal papers, were evaluated. The time period when the extracted papers were published is between 1960 and 2019. The journal papers have been published in a total of 45 journals, where the highest number of papers belongs to the European and Journal of Management Research. Moreover, Management Science, Journal of Optimization Theory and Applications, and Operations Research rank next. Fig. 1 provides the details related to the publication of the extracted papers.

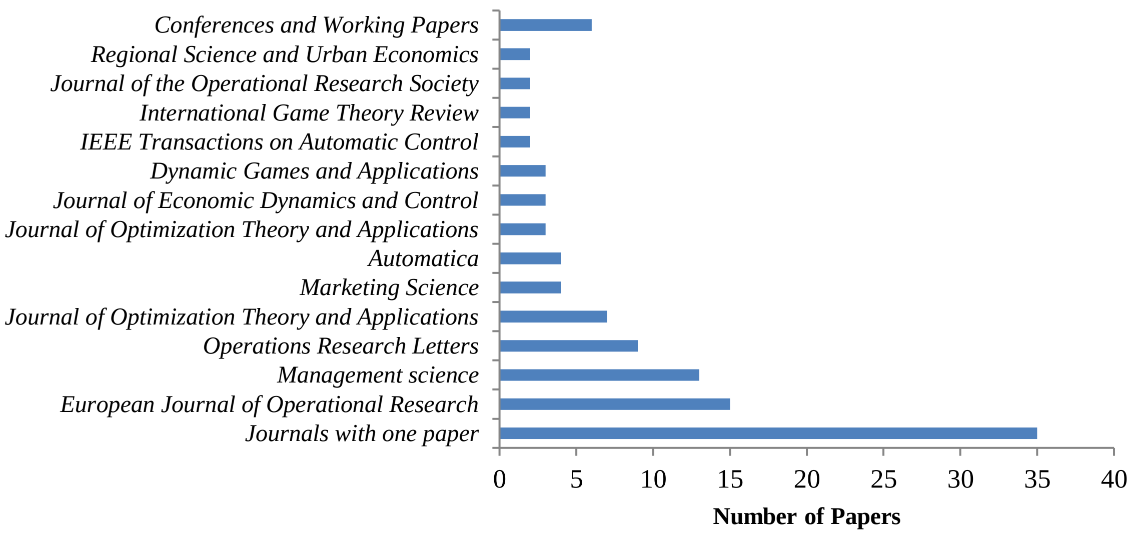

Fig. 1. Number of papers published in the scientific journals

Fig. 2 shows the number of papers published each year from 1960 to 2019. As one can observe, the share of the papers presented in recent years is more than those published in the past years.

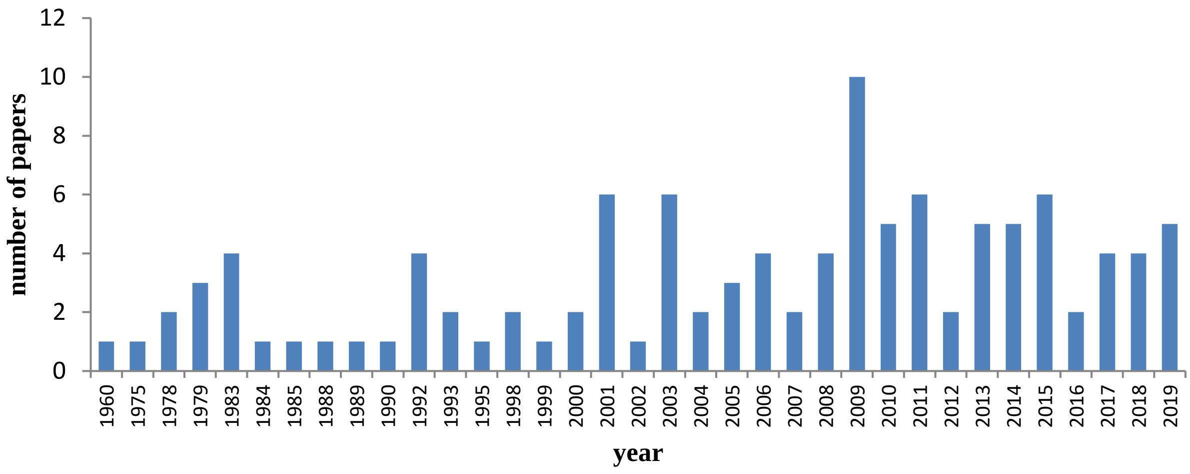

Fig. 2. Number of papers published each year from 1960 to 2019.

This paper aims to identify and discover existing research gaps concerning the considered categorization of the dynamic advertising differential games. To this end, after extracting the papers and collecting them in a datasheet and evaluation by the pivot tables, the papers with more unique features are identified and introduced. In the remaining of the paper in Section 2, the type of categorization for these research works is explained. In Section 3, the analysis results of the papers are studies. Finally, Section 4 provides the conclusion and future recommendations.

2. Advertising differential games

The differential games were first introduced by Isaac [9]. Before the coherent definition of the differential game by Isaac, the concepts of differential games were used in [10] for advertising competition purpose. One of the fundamental components of a differential dynamic game is the differential equation of the model's state variable. Majority of the research in this area is based on four differential models, namely Nerlov-Arrow, (N-A), Vidale-Wolfe (V-W), Lanchester, and diffusion for state variables. Descriptions of the aforementioned models are summarized in Table 2. The explanation of the variables and the parameters given in the fourth column of Table 2 are as follows.

$x_i(t)$ is the market share of the ith company at time t, $A_i(t)$ shows the acquired goodwill caused by the effects of present and past advertising of the ith company at time t, $u_i(t)$ denotes the advertising cost of the ith company at time t, δ_i represents the discount rate of the ith company's goodwill, $\dot{S}(t)$ is the sales rate of the ith company at time t, r_i is the constant amount of the response to advertising cost for the ith company, M represents the market potential, λ_i is the exponential constant of the sale decrease, β_i is the advertising effectiveness of the ith company, ε_i is the word-of-mouth diffusion parameter, and N gives the number of market participants (players).

Table 2. Dynamic advertising models

Model	Researcher	Model development for oligopoly games	Description
Nerlove-Arrow	[11]	[12]	$x_i(t) = \frac{A_i(t)^{\alpha_i}}{\sum_{j=1}^{N} A_i(t)^{\alpha_i}}$ $\dot{A}_i(t) = u_i(t) - \delta_i A_i(t), \quad with\ A_i(0) = A_{i0}, \quad i = 1, \dots, N$
Vidale-Wolfe	[13]	[14]	$\dot{S}(t) = \frac{r_i u_i(t)[M - \sum_{i=1}^{N} S_i(t)]}{M} - \lambda_i S_i(t), \quad i = 1, \dots, N$
Lanchester	[15]	[16]	$\dot{x}_i(t) = \beta_i u_i(t)^{\alpha_i} - x_i \sum_{i=1}^{N} \beta_i u_i(t)^{\alpha_i}, \quad i = 1, \dots, N$
Diffusion	[17]	[18]	$\dot{S}_i(t) = \frac{[r_i u_i(t) + \varepsilon_i S_i][M - \sum_{i=1}^{N} S_i(t)]}{M} - \lambda_i S_i(t), \quad i = 1, \dots, N$

Nevertheless, the main challenge in modeling and solving differential games problems is related to features such as the type of equilibrium (Nash, Stackelberg), solution approach (open-loop, closed-loop, feedback), certainty state (deterministic, stochastic), the number of players (duopoly, oligopoly), and time horizon (finite, infinite). In this paper, in the case there are closed-loop and open-loop solutions for a specific problem in a given paper, the closed-loop solutions are considered (because the closed-loop approach is a more general form of the open-loop approach). This is also considered for duopoly and oligopoly games, and oligopoly games which are the more general form of the problem are taken into account. Table 3 describes all the mentioned types in details.

Table 3. The considered factors for categorizing the papers

Type of category	Sub-categories	Description
Type of equilibrium	Nash	Nash equilibrium (inherited from the name of John Forbes Nash, who proposed it) is a game theory solution that involves two or more players, in which it assumes that each player is aware of the equilibrium strategy of the other players, without any player who can act in a monopoly way to gain profit. If a player selects a strategy, no player can act by changing its strategy while maintaining the other player's profit. Then, the set of current strategy choices and the associated profit form the Nash equilibrium.
	Stackelberg	In general, in a game consisting of a leader who starts the game first, and subsequently, the follower continues the game according to the leader's game, the Stackelberg equilibrium is defined. In the vertical dynamic advertising differential games, the Stackelberg equilibrium is considered for the hierarchy and the decision sequence in marketing channels and supply chains, for instance for the wholesaler and retailer, or franchisor and franchisee.
Solution approach	Open-loop	In an open-loop solution approach for dynamic differential games, it is not assumed that there is dependency between the control variables and the state variable. For example, according to the variables defined in Table 2, $u_i(t, x) = u(t)$. In this case, we cannot change the strategy if we go to a temporal point in the past in the control variable.
	Closed-loop	In a closed-loop solution approach for dynamic differential games, it is assumed that there is dependency between the control variables and the state variable. In

		other words, at any given moment, the control variable is determined considering the value of the state variable.
	Feedback	In the feedback solution approach for dynamic differential games, similar to the closed-loop approach, it is assumed that there is dependency between the control variables and the state variable, except that the control variables are independent of the initial value of the state variable (x_0).
Certainty state	Deterministic	In the deterministic model of differential games, uncontrollable stochastic factors are not considered in the equations of dynamics state.
	Stochastic	In stochastic differential game models, the growth of the state variable is defined based on a stochastic process.
Number of players	Duopoly	In a duopoly game, the game environment is monopolized by two players; in other words, two companies have the full market monopoly.
	Oligopoly	In an oligopoly game, more than two companies have a full market monopoly.
Time horizon	Finite	Players compete in a finite time horizon. In this case, the final conditions and transition to the next period are introduced.
	Infinite	Players compete in an infinite time horizon.

3. A quantitative review

In this section, the extracted papers are studies with regard to the cases considered for categorization such as type of equilibrium, solution approach, number of players, and time horizon. Categorization of the papers has been performed using pivot tables and the obtained results are presented in chart forms. The Appendix presents the reviewed papers along with the details of their features. Fig. 3 illustrates the number of papers with joint paired features. In general, the number of papers addressing the Nash equilibrium is more than that of Stackelberg equilibrium. Regarding the solution approach, feedback, closed-loop, and open-loop approaches have been more studied, respectively. The number of papers in the duopoly state is more than that of the oligopoly state and the infinite time horizon has also been more investigated than the finite time horizon.

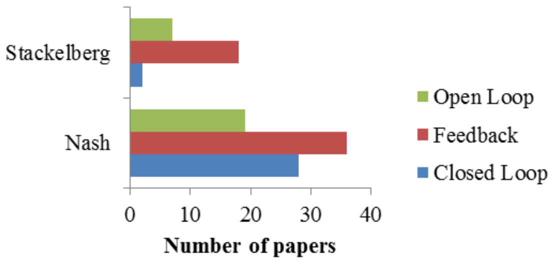

Fig. 3-a. Equilibrium-Solution approach

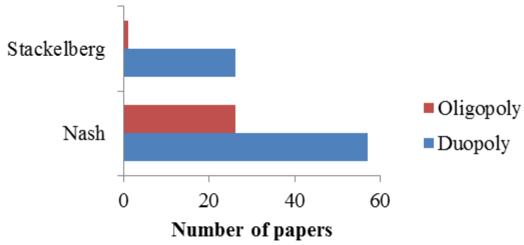

Fig. 3-b. Equilibrium-Number of players

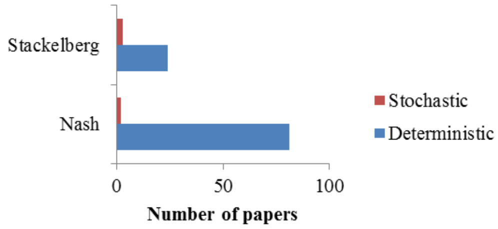

Fig. 3-c. Equilibrium-Certainty

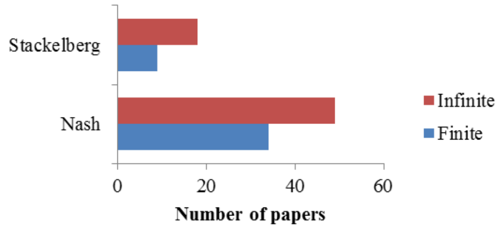

Fig. 3-d. Equilibrium-Time horizon

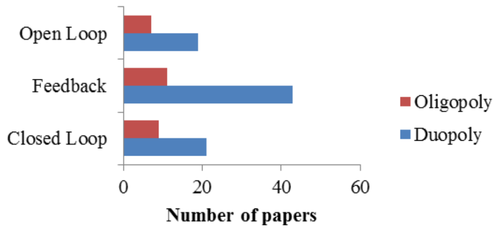

Fig. 3-e. Solution approach-Number of players

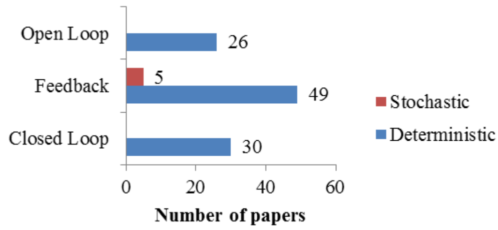

Fig. 3-f. Solution approach-Certainty

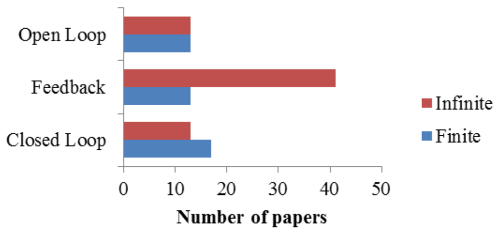

Fig. 3-g. Solution approach-Time horizon

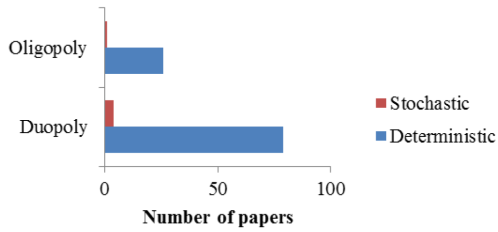

Fig. 3-h. Number of players-Certainty

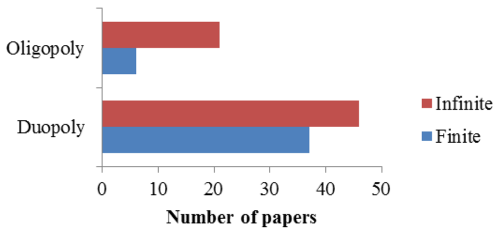

Fig. 3-i. Number of players-Time horizon

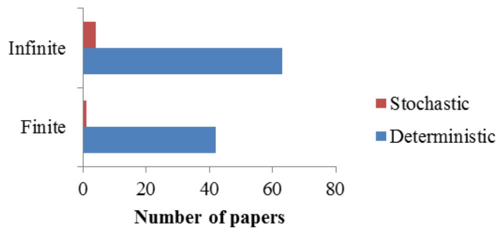

Fig. 3-j. Time horizon-Certainty

Fig. 3. Number of papers with joint paired features

Based on the results given in Fig. 3, the impact of the aforementioned features on the structure and the solution of differential game model is evaluated as follows:

1. As represented in Fig. 3, the number of research on the Stackelberg equilibrium that are mostly modeled in marketing channels in the form of manufacturer/retailer or franchisor/franchisee is less than the same level structures that are often aimed at finding Nash equilibrium points. In addition to Stackelberg equilibrium, the Nash equilibrium approach is also considered for low levels when there is more than a single retailer/franchisee, as in [19, 22].
2. The differences between open-loop, closed-loop, and feedback solution approaches are noticeable [23]. Open-loop strategies are constant over time as they are independent of the system state, meaning that if the player intends to revise its strategy at an intermediate point in the middle of a competition, he/she will not be able to change it. However, in closed-loop strategies, the dependency on the state makes it possible for the players to change their strategies during the competition. If the control variables depend on the initial state of the system, if for any reason, the initial conditions for the players are unknown (the system state at the start point), the equilibrium obtained in the open-loop and the closed-loop will not be a perfect sub-game equilibrium. The feedback strategy is identical to the closed-loop strategy except that the control variable is independent of the initial state of the system and thus, it can provide a perfect sub-game equilibrium [2]. Generally speaking, the computational challenges in the open-loop approach are less than the other two approaches [24]. In differential games, in contrast to optimal control problems that include a decision-maker (player), there is a system of partial differential equations (PDEs) with computational complexity even at the best condition, while these strategies are more adapted with practical conditions [2]. Thanks to better adaptation to the practical conditions, scholars normally attempt to avoid solution challenges of the closed-loop and feedback models and prefer to use this solution approach. For instance, for the duopoly market conditions, the authors of [24] analyzed the equilibrium strategies as a function of a state variable in a static state with no direct dependency on time. In this method, PDEs are transformed into ordinary differential equations (ODEs). For the cases with more than two players with dependency on more than one state variable, and/or when there is more than one state variable for the system (e.g., market share and quality [25, 26]), a nonlinear system of PDEs appear, where the number of equations is less than the number of decision variables (control) and it is practically impossible to solve the problem [25]. It might be said that this is the reason the oligopoly competition has rarely been considered by scholars. Nonetheless, approaches such as dynamic conjecture variability have been proposed which is based on the players' conjecture of control variable changes with respect to the system state variations [2], resulting in a reduced number of variables for differential equations.
3. In general, the dynamic scheduling concepts such as optimality principle, Bellman and Hamilton-Jacobi-Bellman equations, and optimal control concepts such as Pontryagin’s maximum principle are utilized, in general, to solve differential games. These methods provide the necessary and sometimes sufficient conditions for the problem optimality. The limitation of the time horizon adds constraints of the final conditions and transition to the next time horizon, complicating the solution of the problem [27]. As a result, researchers have often focused on problem modeling in infinite conditions.

4. Review of key papers

Since the focus of this study is on finding research possibilities in the area of advertising differential games, those papers have been studied as key papers for development that there were a maximum of two similar papers with them related to the standard joint features (for example equilibrium-solution approach). Studies in [28], [29], [29], [30], [31] have more than one monopoly in the joint paired feature. Thus, a total of seven researches have more unique features than other researches. Table 4 lists the researches with maximum two similar researchers at each equilibrium type feature category. These papers are studies as follows.

Table 4. Key papers to study

Feature/Research		[32]	[33]	[29]	[28]	[34]	[30]	[31]
Stackelberg	Closed-loop	*	*					
Stackelberg	Stochastic			*	*			
Stackelberg	Oligopoly					*		
Nash	Stochastic						*	*
Open-loop	Stochastic	-	-	-	-	-	-	-
Closed-loop	Stochastic	-	-	-	-	-	-	-
Oligopoly	Stochastic				*			
Duopoly	Stochastic			*			*	*
Finite horizon	Stochastic							*
Infinite horizon	Stochastic			*	*		*	

First, the papers with the Stackelberg equilibrium and then papers dealing with the closed-loop solution approaches are investigated. Ref. [32] has studied the dynamic advertising equilibrium strategies with symmetric information structure and sequential game (ordinal) by considering a leader and a follower (duopoly conditions). In this context, the leader's market share is defined by (3):

$$\dot{M} = (1 - M_t)C_l(a_{lt}) - M_t C_f(a_{ft}), \quad t \in [0, \infty), M_0 \text{ given} \tag{3}$$

Where M_t is the leader's market share at time t, a_{it} is the advertising cost of the player $i, i \in \{l(\text{leader}), f(\text{follower})\}$, and $C_i(.)$ is the concave positive incremental function that determines the impacts of advertising cost on the market share. Moreover, the discount profit for the leader (π_l) and the follower (π_f) is defined as follows:

$$\pi_l = \int_{t=0}^{\infty} e^{-rt}(g_l m_t - a_{lt}), \tag{4}$$

$$\pi_f = \int_{t=0}^{\infty} e^{-rt}\big(g_f(1 - m_t) - a_{ft}\big) \tag{5}$$

In the above equations, g_i represents the marginal net profit that the ith competitor will achieve in a time period at each point in the market share, and r_i is the discount rate for the ith competitor. Ref. [32] addresses the proposed model using the concepts of dynamic scheduling value function by presenting a numerical solution approach according to [35] and compared the results of the closed-loop and open-loop solution cases. The evaluation results of the model proved that there is no difference between the leader and follower strategies at an infinite time horizon and, according to real data (Coca Cola and Pepsi Companies), the Stackelberg conditions have better described the current situation than the Nash conditions. The oligopoly development capability of the follower in the model and considering the finite time horizon can be proposed for future studies. Based on the modified Nerlov-Arrow model, the authors in [33] considered a model for a manufacturer and a retailer as a Stackelberg game, where the goodwill variations for the manufacturer and the retailer are developed using (6) and (7), respectively.

$$\dot{G}_M(t) = -\delta G_M(t) + \alpha_M U_M(t - d_M), \;\; G_M(0) = G_{M0} \tag{6}$$

$$\dot{G}_R(t) = -\delta G_R(t) + \alpha_R U_R(t - d_R), \;\; G_R(0) = G_{R0} \tag{7}$$

In the decentralized case, the performance index for the manufacturer and retailer is defined as:

$$J_M = \int_0^{\infty} e^{-rt}\{\alpha S(t) - C(U_M(t))\}dt \tag{8}$$

$$J_R = \int_0^{\infty} e^{-rt}\{(1-\alpha)S(t) - C(U_R(t))\}dt \tag{9}$$

And in the decentralized case:

$$J_{MR} = \int_0^{\infty} e^{-\lambda t}\{S(t) - C\big(U_M(t)\big) - C\big(U_R(t)\big)\}\,dt \tag{10}$$

In Eqs. (6) to (10), $G_M(t)$ and $G_R(t)$ are the goodwill of the brand for manufacturer and retailer, U_M and U_R denote the advertising cost for the manufacturer and retailer, δ is the discount rate (representing the forgetfulness effect of the brand), α_M and α_R are the effectiveness factor of advertising on the goodwill of the manufacturer and retailer, d_M and d_R are the delay levels of the effectiveness of advertising on the customer (time duration from receiving advertising information to product purchase) for the manufacturer and retailer, J_M and J_R show the performance index or the profit of manufacturer and retailer in the decentralized case, J_{MR} is the objective function of the entire channel, S(t) is the demand function, and $C(U_M(t))$ and $C(U_R(t))$ are the advertising cost. Using the maximum principle of Pontryagin condition, ref. [33] has analytically obtained the values of the decision variable and showed that the equilibrium cost in the centralized case is more than the decentralized system. It is possible to develop the model in the oligopoly game considering a finite time horizon.

As mentioned before, one of the differential game modeling challenges is the number of players participating in the system. This is challenging in models that the dynamics of the competitors are interdependent. Ref. [34] has developed a model in a structure with single manufacturer/multi-retailer under the cooperative condition in which the manufacturer is committed to providing a portion of the advertising cost of the retailer based on a specified subsidy rate, θ_i. In this case, in a Stackelberg game, the manufacturer initially announces its subsidy rate and retailers as the follower in an oligopoly market find the Nash equilibrium to maximize the cumulative sale and select the optimal advertising effort. The modified diffusion model is considered as the base model for the state variable growth of the retailer [36]:

$$\dot{X}_i(t) = \rho_i u_i(t) D_i \sqrt{1 - \bar{X}(t)}, \qquad X_i(0) = X_i \in [0,1], i = 1,2,\dots,n \tag{11}$$

Where X_i is the cumulative sale of the ith retailer, $u_i(t)$ denotes the advertising effort rate of the ith retailer at time t, ρ_i shows the effectiveness of the ith retailer's advertising, D_i is the product demand of the ith retailer, and $\overline{X}(t) = \sum_{j=1}^{N} X_i(t)$ gives the total cumulative sale of N retailers. In this game, after determining the subsidy rate by the manufacturer, the retailers maximize the present value function of their profits using (12):

$$J_i = \int_0^{\infty} e^{-rt}\left((p_i - w_i)\dot{X}_i(t) - \left(1 - \theta_i\big(X(t)\big)\right)u_i^2(t)\right)dt, \quad i = 1,2,\dots,n \tag{12}$$

In (12), p_i is the retail price of the ith retailer, w_i is the wholesale price for the ith retailer, and r represents the discount rate. The manufacturer maximizes the following objective function by predicting the retailer's response to find the optimal subsidies strategy:

$$J_M = \int_0^{\infty} e^{-rt}\sum_{j=1}^{n}\left[w_j\dot{X}_j(t) - \theta_j(t)\left[u_j\big(X(t)|\theta_1(t),\dots,\theta_n(t)\big)\right]^2\right]dt \tag{13}$$

Authors in [34], analytically and using the Hamilton-Jacobi-Bellman equation, defined the optimality condition of the problem in the feedback state and showed the impact of the dependency of subsidy rates on the determining model parameters on the channel's profit. The notable point in solving this problem is using dynamic scheduling concepts that make it possible to confront an oligopoly state. Furthermore, the concave value function in the Hamilton-Jacobi-Bellman equation and the satisfaction of the second-order optimality conditions are other features of the problem modeling. It is possible to solve the model in a stochastic environment for future research. A stochastic environment in differential games is generally modeled as a stochastic process considering the state variable. So far, the state variable has been defined either as a geometric Brownian motion with the inclusion of the white noise perturbation factor and the state variable variance in the dynamic state equation or presented as an Ito equation [28-30]. Moreover, in [31], the stochastic goodwill process in the Nerlov-Arrow model considered as a Poisson process. These equations make the optimization computation become stochastic either due to using stochastic calculus or differential equations methods. In general, in the literature of advertising differential games, there are only two stochastic research studies concerning the Nash-based problem modeling. Additionally, there are merely two research studies in this regard for the Stackelberg approach. This subsection addresses these studies. As previously been noted, ref. [31] has developed the Nerlov-Arrow model in a duopoly state under uncertainty considering the Poisson process for the goodwill state variable G. The problem modeling is given in (14):

$$\begin{aligned} &Maximize \quad J_i = E\int_0^{T} e^{-rt}\left[\pi_i(x_i) - w_i(a_i)\right]dt = \int_0^{T} e^{-rt}\left[h_i\left(\frac{SG_i}{(G_1+G_2)}\right) - w_i(a_i)\right]dt, \quad i = 1,2 \\ &S.t. \\ &\qquad \dot{G}_i = -\delta G_i + a_i, \quad G_i(0) = G_i^0, \quad i = 1,2 \end{aligned} \tag{14}$$

In Eq. (14), x_i represents the market share of the ith company, $\pi_i(x_i)$ denotes the profit function, a_i is the advertising cost, $w_i(a_i)$ is the advertising cost function, S is the Poisson process parameter, and $h_i(.)$ is a computable function with regard to the mathematical expectation of the profit function. The mentioned model was analyzed in [31] and using the maximum of the optimality principle an open-loop solution for the problem was proposed. Furthermore, it was suggested that the model presented in this study is developed in an oligopoly mode or Stackelberg structures. Ref. [30] has developed a duopoly Lanchester model in a stochastic manner. The problem formulation is a combination of the formulae given in [37] and [38] and is expanded as Eq. (15):

$$\begin{aligned} &Maximize \quad J_1 = E\int_0^{\infty} e^{-r_1 t}[m_1 x(t) - c_1 u_1(t)^2]dt \\ &Maximize \quad J_2 = E\int_0^{\infty} e^{-r_2 t}\left[m_2\big(1-x(t)\big) - c_2 u_2(t)^2\right]dt \\ &S.t. \\ &\dot{x} = \left(\rho_1 u_1(x)\sqrt{1-x(t)} - \rho_2 u_2(x)\sqrt{x} - \delta(2x-1)\right)dt + \sigma(x)dz(t), \quad \mathrm{x}(0) = \mathrm{x} \\ &\in [0,1], \qquad \mathrm{t} \ge 0 \end{aligned} \tag{15}$$

To solve the model, ref. [30] employed the Hamilton-Jacobi-Bellman equation and estimated the sufficient and necessary conditions of optimality. Moreover, the model was solved in both symmetric (equality of parameters of two companies) and asymmetric (inequality of parameters of two companies) modes. The model development in the oligopoly state besides the use of price decision variables for dynamic market control was recommended for future study. Authors in [28] developed the model given in [37], a particular type of Vidale-Wolfe model known as the traditional model, where the dynamics state equation for the retailer is defined as (16):

$$dx(t) = \left(\rho u(t)\sqrt{1-x(t)} - \delta x(t)\right)dt + \sigma\big(x(t)\big)dz(t), \;\; x(0) = x \in [0,1], \qquad t \ge 0 \tag{16}$$

where $\mathrm{x(t)}$ shows the awareness share, representing the ratio between the number of customers aware of the product by advertising efforts, $\mathrm{u(t)}$ is the advertising effort, ρ is the response to advertising at time t, $\delta(\mathrm{t})$ is the discount rate (because of existing competitive background, product obsolescence, forgetfulness and etc.). In the second term of the right side of (16), $\sigma\big(\mathrm{x(t)}\big)$ denotes variance and $\mathrm{z(t)}$ is the standard Wiener process under a given probability space. The sequence of events in the model is such that the manufacturer, initially, determines the wholesale price, $\mathrm{w(x)}$, and the participation rate, $\theta(\mathrm{x})$, and then the retailer specifies the retail, $\mathrm{p(t)}$, and the advertising effort, $\mathrm{u(t)}$, to give an optimal response to the manufacturer strategy based on the objective function (17):

$$\mathrm{J_R} = \mathrm{E}\int_0^{\infty} \mathrm{e}^{-\mathrm{rt}}\left\{\left(\mathrm{p(t)} - \mathrm{w}\big(\mathrm{x(t)}\big)\mathrm{D}\big(\mathrm{p(t)}\big) - \left(1 - \theta\big(\mathrm{x(t)}\big)\right)\mathrm{u(t)^2}\right)\right\}\mathrm{dt} \tag{17}$$

The manufacturer predicts the retailer's reaction functions and utilizes them in his optimal control problem (Eq. 18) to achieve the optimal wholesale price w(x) and the participation rate θ(x):

$$\begin{aligned} &Maximize \quad \mathrm{J_M(x)} = \mathrm{E}\int_0^{\infty} \mathrm{e}^{-\mathrm{rt}}\Big\{(\mathrm{w(t)} - \mathrm{c})\mathrm{D}\big(\mathrm{p}(\mathrm{x(t)}|\mathrm{w(t)}, \theta(\mathrm{t}))\big)\mathrm{x(t)} \\ &\qquad - \theta(\mathrm{t})\big(\mathrm{u}(\mathrm{x(t)}|\mathrm{w(t)}, \theta(\mathrm{t}))\big)^2\Big\}\mathrm{dt} \\ &S.t. \end{aligned} \tag{18}$$

$$dx(t) = \left(\rho u(x(t)|w(t), \theta(t)\sqrt{1-x(t)} - \delta x(t)\right)dt + \sigma\left(x(t)\right)dz(t), \quad x(0) = x \in [0,1], \quad t \geq 0$$

Authors in [28] used the essential optimality condition of the Hamilton-Jacobi-Bellman equations and presented an open-loop solution to the problem. Also, ref. [29] has developed the model given in [28] as such that it applies the direct effectiveness of the manufacturer on cooperative advertising in the problem model. In this case, when the retailer focuses on public advertising, the manufacturer allocates subsidies to retailers and participate in general advertising. Hence, the retailer's dynamic state equation is modified as Eq. (19), where $u_R(t)$ represents the retailer's advertising efforts and $u_M(t)$ is the advertising efforts of the manufacturer.

$$dx(t) = \left(\alpha\left(u_R(t) + u_M(t)\right)\sqrt{1-x(t)} - \delta x(t)\right)dt + \sigma\left(x(t)\right)dz(t), \quad x(0) = x \in [0,1], \quad t \geq 0 \qquad (19)$$

The retailer optimizes the objective function (20) with respect to the constraint (19) and noting that u_M and the participation rate θ are determined by the manufacturer in the first step, consequently the optimal strategy u_R is obtained.

$$J_R = E\int_0^{\infty} e^{-rt}\{mx(t) - u_R(t)^2 + \theta(t)u_R(t)^2\}dt \qquad (20)$$

Similar to [28], the manufacturer solves his optimal control problem to find optimal values of u_M and θ (Eq. 21):

$$\begin{aligned} &Maximize \quad J_M = E\int_0^{\infty} e^{-rt}\{Mx(t) - \theta(t)u_R(x(t)|\theta(t), u_M)^2 - u_M(t)^2\}dt \\ &S.t. \\ &dx(t) = \left(\alpha(u_R|\theta(t), u_M(t)) + u_M(t))\sqrt{1-x(t)} - \delta x(t)\right)dt + \sigma\left(x(t)\right)dz(t), \quad x(0) = x \in [0,1], \quad t \geq 0 \end{aligned} \qquad (21)$$

It is possible to develop models proposed in [28] and [29] in the oligopoly structure on the follower side with the finite time horizon.

5. Conclusions

This paper addresses the literature review on the dynamic differential games. To realize this, considering the features of differential games including the type of equilibrium (Nash and Stackelberg), the solution approach (open-loop, closed-loop, and feedback), certainty (deterministic and stochastic), the number of players (duopoly and oligopoly), and the time horizon (finite and infinite) as well as utilizing the pivot table method, the number of papers dealing with each feature were studied. Based on the obtained results, in a general evaluation, the relationships between features in terms of the number of papers with that feature are provided in Table 4. Taking into account the results and analyzing the papers focusing on different features, the possibility and simplicity of solving are the most important reasons for prioritizing the type of equilibrium, certainty, number of players, and time horizon features and adapting with the real conditions is the main reason for prioritizing the type of equilibrium feature. Furthermore, in the rest of the paper, by extracting and describing seven key papers with unique paired features compared to other papers in this realm, we proposed the development facilities to expand such papers. In this study, the internal features of the models such as the type of dynamics state equation, cooperativeness/uncooperativeness game, and symmetric/asymmetric game are not considered as the discriminative features. Additionally, the solution approaches were not analyzed in detail. For future studies we can focus on the mentioned features and analytical approaches for solving the presented models.

Appendix

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[10]		●		●			●			●			●					●						●		1960
[34]		●	●				●		●	●			●					●						●		1975
[35]		●	●				●	●		●			●					●						●		1978
[36]		●	●				●			●			●					●						●		1978
[29]					●	market potential			●		●			●				●						●	·	1979
[37]		●	●				●			●			●					●						●		1979
[38]		●	●				●	●		●			●					●						●		1983
[18]		●	●				●			●					●				●						●	1983
[39]		●	●				●			●						●		●						●		1983
[40]		●			●		●			●				●					●						●	1983
[12]					●		●							●					●					●	·	1984
[41]		●	●				●	●		●		●	●					●						●		1985
[42]					●		●				●			●					●					●	·	1986
[43]		●	●					●		●					●			●							●	1988
[33]		●		●			●		●	●		●						●						●		1989
[44]					●		●							●				●						●	·	1990
[45]		●			●		●		●	●				●				●		1	1				●	1992

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[46]		●	●				●			●					●				●						●	1992
[47]					●		●	●						●				●						●	·	1993
[48]		●		●			●	●		●		●							●						●	1993
[49]		●						●		●		●						●							●	1995
[16]		●		●			●	●		●		●							●						●	1998
[50]		●		●			●		●	●			●					●							●	1998
[51]		●			●				●	●				●				●		1	1			●		1999
[52]		●			●				●	●				●				●		1	1				●	2000
[53]		●				price			●	●						●		●						●		2000
[54]	●	●			●				●	●				●				●		1	1				●	2001
[55]	●	●			●				●	●				●				●		1	1				●	2001
[56]		●		●			●	●		●		●						●						●		2001
[57]		●		●					●	●		●						●							●	2001
[58]		●	●					●						●					●						●	2002
[59]		●				reservation price	●		●	●				●					●						●	2003
[60]	●	●			●				●	●				●				●		1	1				●	2003

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[61]		●			●	cumulative promotion			●	●				●					●	1	n			●		2003
[62]		●			●				●					●				●		1	1			●		2003
[63]	●		●						●	●						●		●							●	2003
[28]		●		●					●				●					●							●	2004
[30]				●					●	●		●						●							●	2004
[64]	●						●			●				●					●	2	1				●	2005
[65]		●	●					●		●		●						●							●	2005
[65]		●	●					●		●		●						●						●		2005
[66]		●			●		●			●				●				●							●	2006
[23]	●			●			●	●		●		●						●						●		2006
[66]		●						●		●		●						●						●		2006
[67]		●		●				●		●		●						●						●		2006
[68]		●		●			●	●		●		●							●	1	1				●	2007
[69]		●	●				●	●		●		●						●						●		2007
[70]		●			●				●	●				●				●		1	1				●	2008
[71]	●				●				●	●				●				●		1	1				●	2008
[20]	●						●			●			●							1	1			●		2008
[72]		●		●			●	●		●		●							●						●	2008
[73]	●		●		●		●			●				●				●						●		2009

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[74]		●			●		●			●				●				●							●	2009
[75]	●	●			●				●	●				●				●		1	1				●	2009
[76]	●				●				●	●				●					●			1	2		●	2009
[77]		●		●					●				●						●						●	2009
[78]		●		●					●				●						●						●	2009
[26]		●		●					●		●		●						●	1	2				●	2009
[79]		●	●				●	●				●							●						●	2009
[80]		●				price	●			●						●			●						●	2009
[81]		●					●		●					●				●							●	2010
[82]		●		●					●				●					●							●	2010
[83]		●		●					●				●						●	1	2				●	2010
[84]		●	●				●	●		●						●		●						●		2010
[81]		●			●				●	●				●				●							●	2010
[85]	●				●				●	●				●				●		1	1				●	2011
[86]	●				●				●	●				●					●			1	2		●	2011
[87]		●		●					●				●					●		1	1				●	2011
[21]		●			●	backlog			●	●			●						●					●		2011
[88]		●			●				●	●				●				●							●	2011
[٨٩]		●		●					●		●		●						●						●	2012
[90]		●		●				●				●						●							●	2012

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[91]	●				●	Reference price	●			●				●				●		1	1			●		2013
[۹۲]		●	●				●			●			●					●		۱	۲			●		2013
[۹۳]		●		●			●				●	●	●					●							●	2013
[۹۱]	●				●		●			●				●				●							●	2013
[۹۴]		●		●				●					●					●							●	2013
[95]		●					●			●		●						●						●		2014
[27]	●		●						●	●					●			●		1	1			●		2014
[96]		●	●		●		●			●				●				●							●	2014
[۹۷]		●		●					●	●		●						●		۱	۱				●	2014
[98]		●			●	Inventory Level			●	●				●					●					●		2015
[99]		●	●				●			●		●							●					●		2015
[100]		●	●					●		●		●						●						●		2015
[۱۰۱]		●			●			●		●				●					●						●	2015
[۹۹]		●	●				●						●						●						●	2015
[۱۰۲]		●	●				●	●		●		●							●					●		2015
[24]	●							●						●				●		1	1			●		2016
[۱۰۳]		●	●				●			●					●			●							●	2016
[۱۰۴]	●				●		●			●				●				●		۱	۱				●	2017
[۲۵]	●			●					●		●		●					●		۱	۱				●	2017
[105]		●	●				●			●			●					●						●		2017

References	Equilibrium		State Variable				Solution Approach			Uncertainty		Base Model					Number of Players			Supply Chain Characteristics				Horizon		year
	Stackelberg	Nash	Sales	Market Share	Goodwill	Other	Open Loop	Closed -loop	Feedback	Deterministic	Stochastic	Lanchester	Vidale-Wolf	Nerlov-Arrow	Diffusion	Other	Monopoly	Duopoly	Oligopoly	Manufacturer	Retailer	Franchisor	Franchisee	Finite	Infinite	
[۱۰۶]	●		●						●		●							●		1	1				●	2018
[۱۰۷]	●			●					●		●		●					●		1	1				●	2018
[۱۰۸]	●	●			●				●		●				●			●		1	1				●	2018
[۱۰۹]		●		●				●			●		●						●					●		2018
[۱۱۰]		●		●		demand	●	●	●		●		●						●						●	2019
[۱۱۱]		●			●		●				●				●			●						●		2019
[۱۱۲]					●					●	●				●			●						●		2019
[۱۱۳]	●	●		●					●		●		●					●		1	1				●	2019
[۱۱۳]	●	●		●					●			●	●					●		1	1				●	2019

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