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The Impact of Demand Dependence on Optimal Inventory Level and Pooling Effect

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ABSTRACT

In supply chain inventory management, it is generally approved that inventory pooling is a suitable practice and studied its impact on expected profits under restrictive assumptions on demands distribution (Demands are often assumed to be independently and identically distributed and whenever demands dependence is included, it has been assuming bivariate or multivariate normal distribution). In this paper, we investigate the impact of pooling on stocks levels in addition to the impact of dependence on stocks levels when demand distributions do not follow the normality assumption. In addition, we analyze the impact of marginal distribution of demand (identical or no-identical) and distribution demand with different Skewness on the inventory level. Furthermore, we prove that the dependence structure, the coefficient of dependence, and the distribution of the local demands affect the value of the benefit of pooling using various copulas. Finally, the conditions under which the pooling is preferred to the no pooling case are likewise affected.

1. Introduction

Inventory management is a main factor in coping with several uncertainties can be occur in supply chain and consequently preventing stock-outs and guaranteeing delivery to customers with minimal delays and interruptions [15]. One of the most important uncertainties is that of demand [3], in effect, demand uncertainty depicts one of the overriding sources of risks in supply chain management. Inventory pooling is used to deal with this risk and therefore to decrease inventory cost. Generally, savings occurring from pooling are defined based on an independent system where each retailer is disserved exclusively by a manufacturer [7].

Most studies found in the literature dealing with inventory pooling problem focus on its impact on expected costs or profits [6, 7, 16 and 18]. It's more interesting to study also the impact of inventory pooling problem on optimal stocks levels in today's businesses. Some authors [1, 5, 8, 9, 10, and 11] have analyzed this effect but under restrictive assumptions on demands distribution. Indeed, whenever demands dependence is included, it has been assuming bivariate or multivariate normal distribution. However, upon searching the studies related to the inventory pooling problem, there is no formal mechanism to assess the impact of dependence on stock level when demand distributions do not follow the normality assumption.

Other previous studies have treated particular cases where demands are modeled by multivariate distributions. Demands are often assumed to be independently and identically distributed (i.i.d). By way of example, Benjaafar et al. [4] have assumed that demands are i.i.d random variables following a Poisson distribution. Yang and Schrage [19] have assumed that demands are i.i.d following an asymmetric right skewed distribution. In reality, product demands are neither independent nor identically distributed. Demand modeling in multivariate framework allows taking into account at the same time the distribution of each demand and the dependence structure between two-by-two demands.

Assuming a demand vector that follows a multivariate normal distribution means that each individual demand follows a

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univariate normal distribution and that any demand can be expressed as a linear function of all other demands and residuals of a regression that follow a Gaussian distribution. Assuming a demand vector that follows a multivariate student distribution for example means not only that each individual demand follows a univariate student distribution with the same degree of freedom but that joint distribution can be modeled by student copula. Each of these assumptions are rarely verified in practice. by limiting oneself to the usual multivariate distributions, there is a risk of modeling the vectors of the demands by multivariate distribution that is not appropriate. This may affect the correct determination of the optimal inventory level. We therefore need a powerful mathematical tool giving greater flexibility in the modeling of multivariate data. In this context, the copula approach provides a solution of these issues and offer solution to the problem of the violation of i.i.d demands hypothesis and demands that follow usual multivariate distribution. This approach allows considering non-normal dependent demands with different (non-linear) dependence structures among the demands.

In addition, many studies in various domains such as finance and insurance show that modeling by multivariate distribution pass necessary by modeling adequately marginal distribution before the selection of the most adequate copula. Respect of previous steps can provide a good estimate of the quantile of multivariate distribution used to model demands vector[1].

Our contribution in this paper is to study the impact of dependence structure between demands (characterized both by specific copula having well-defined properties and the Kendall tau) and degree of asymmetry of marginal demand distribution on optimal inventory level and pooling benefits. We determinate conditions under which pooling can be an optimal strategy, neutral or not at all beneficial.

The aim of this paper is to consider the non-normal marginals distributions and the copula to join them, i.e., we discuss the impact of pooling and different structure dependence on the optimal inventory level. Further, we include sensitivity analyses of dependence structure at choice of marginal distributions and the sensitivity of the results with respect to Kendall tau (dependence measure). The percentage pooling effect on inventory levels under different copulas corresponding to different values of Kendall's is also considered. In this paper, we use different copulas functions with different values of Kendall's (different parameters values) and with different marginal distributions.

The outline of this paper is as follows. A pooling inventory model is provided in section 2. In section 3 we discuss analytically the effect of the dependence structure on the inventory level. Finally, section 4 concludes the paper.

2. Pooling inventory model

2.1. Method for determining the inventory level

To discuss the benefit of demand pooling in terms of optimal inventory levels, a model inspired by Aydin et al. [2] is considered:

c: the holding cost for each unit.

p: the profit for each demand satisfying from inventory.

The unmet demand is lost along with the overstock items.

The objective is to determine the stock level Q maximizing the expected total profit. Additionally, the optimal inventory level is a quantile of the demand distribution of order $\alpha = t = \frac{p-c}{p} = 1 - \frac{c}{p}$ as showed in Equation (1):

$$F^{-1}\left(\frac{p-c}{p}\right) = \underset{Q}{argmax}\{pE_D[min(D,Q)] - cQ\} \qquad (1)$$

where $F(.)$ is the demand distribution function.

This quantity can be interpreted as the threshold of demand that will be exceeded with a probability of 1-α. The demand does not exceed this threshold t with a probability α.

In inventory pooling problem, it is generally considered two identical items with uncertain demands D_1 and D_2. These two products have the same unit profit as well as the unit holding cost. There are two alternatives to satisfy the demand: holding dedicated inventory of each product, or keeping a one inventory for the aggregate demand $(D_1 + D_2)$. It has been proven that keeping a single inventory (pooling) can be the best option for determining the optimal inventory levels.

In the first alternative, the optimal inventory level can be determined via $F_{D_1}^{-1}(t) + F_{D_2}^{-1}(t)$ where F_{D_i} is the individual distribution function of D_i (i=1,2) and $t = \frac{p-c}{p}$ is considered as the margin ratio. It has been noticed that this quantity of inventory does not take into account the dependence structure between demands. Alternatively, the optimal pooled inventory level $F_{D_1+D_2}^{-1}(t)$ depends not only on the marginal demand distributions of D_1 and D_2, but also on the dependence structure between D_1 and D_2. In practice, pooling is the best option, but it is necessarily known whether to increase or reduce inventory as a result of that decision. If consolidation needs higher levels of total inventory, then pooling effect is positive. Likewise, pooling effect is negative when pooled inventory level is inferior than the dedicated inventory. The pooling effect can be determine as following in Equation (2):

$$F_{D_1+D_2}^{-1}(t) - F_{D_1}^{-1}(t) - F_{D_2}^{-1}(t) \qquad (2)$$

2.2. cooling effect measurement

In this section, we interested to the impact of margin ratio on the sign of pooling effect.

Proposition 1

Let $PE(t) = F_{D_1+D_2}^{-1}(t) - (F_{D_1}^{-1}(t) + F_{D_2}^{-1}(t))$ and assuming that the equation $PE(t) = 0$ admits a unique solution t_0 in [0, 1]. Thereby, t_0 is a threshold such that the consolidation effect is less than zero if and only if $t > t_0$.

From this proposition, we can conclude that if $PE(t) = 0$ admits a unique solution t_0 in the interval [0,1], then the consolidation effect can only change from a negative sign to a

[1] In finance and insurance domains, a popular technic frequently used to estimate portfolio risk after 1994 is the Value at Risk (VaR) that is defined as the quantile of multivariate distribution [12 and 13].This risk measurement tool plays the same rule by analogy as optimal inventory level in operational and supply chain management contexts.

positive sign once t increases. The threshold t_0 depends both on the marginal distributions and the dependence structure between the joined distributions. If the demand vector follows a multivariate normal distribution, the critical threshold is equal to 0.50 regardless of the parameters of this distribution. The common normality assumption results in a symmetry between the positive effect and the negative effect of pooling. If $t > 0.50$, pooling allows having a lower inventory level than in no pooling system and vice versa. In the case where t_0 is different from 0.50, there is an asymmetry between the positive and the negative pooling effect.

Proposition 2

When demand (D_1 and D_2) distributions follow the elliptical family (such as Normal, Student, Cauchy, Logistic distributions…), the consolidation strategy will result in a lower inventory if and only if the marginal ratio is greater than 0.50.

A distribution function F is regularly varying at $-\infty$ with tail index $\alpha > 0$ if the following condition presented in Equation (3) is verified:

$$\lim_{t\to\infty}\frac{F(-tx)}{F(-t)} = x^{-\alpha}, \quad \forall x > 0 \qquad (3)$$

Proposition 3

We suppose that the tail index of the common demand distribution is insignificant related to its value for the individual distributions.

Let D_1 and D_2 be identically distributed random variables with distribution functions which vary regularly with the same tail index $\alpha > 0$.

There is a threshold such that the marginal ratio t is greater than or equal to t_0. In this case, the effect of consolidation is positive if $0 < \alpha < 1$ and it is negative if $\alpha > 1$.

This proposition shows that when the demand distribution varies regularly the sign of the pooling effect depends on the properties of the tails of the demand distribution.

It is possible in certain situations to encounter a zero threshold. In this case, pooling will result in a higher inventory level with any level of the marginal ratio. This is the example of identically and independently distributed demands according to a Pareto distribution.

These findings will be illustrated in our numerical analysis. Moreover, the impact of dependence structure and the chosen copula are analysed on the inventory level. The impact of the marginal demand distribution and the Kendall's tau variation are also examined on the inventory level.

3. Analytical example explaining the effect of the dependence structure on the inventory level

To gain additional insights into the inventory pooling problem, we have performed a numerical analysis under the newsvendor framework. Especially, we are interested when product inventories are pooled, how inventory levels are changed, and how (Symmetric (null skewness) versus asymmetric (Negative or positive skewness) and identical versus no identical) marginal demand distributions and dependence structure affect the inventory pooling decisions. Therefore, the steps taken to achieve this objective are:

(1) To determine the inventory levels under different demands distributions with different skewness.

(2) To compare between inventory levels whether the two distributions are identical or nonidentical.

(3) To perform a comparison across the different copulas to give interesting insights into seeing the effect of dependence structure on inventory levels.

3.1. Copulas and distributions used to construct bivariate data

Copula theory, first introduced in 1959 by Sklar [17], is a very powerful statistical tool allowing greater flexibility in the modeling of multivariate data. The objective of copulas is to decompose a joint distribution into two elements: the marginal distributions on the one hand and a mathematical function which makes the connection between them by modeling their dependencies. This makes it possible to extend certain results obtained in a univariate framework to the multivariate case. The multidimensional distributions thus obtained are more general and are more in line with reality. Copula functions describe how individual marginal distributions are coupled together by a joint distribution.

Even if the distributions of the marginals are different, it becomes possible to determine a multivariate distribution for these data by using the copula functions. The principle of this theory consists in the first place to model each series by a usual univariate distribution, to estimate the parameters of this distribution then to transform the data to the uniform [0,1] using the inverse of the distribution function. Second, one must choose from a set of copula functions the one that best describes the dependence structure. In this way, a multivariate distribution is characterized not only by the distributions of the marginals but also by the copula which describes the structure of dependence between these distributions. The multidimensional distributions thus obtained better reflect reality.

Thus, a two-dimensional copula C is a cumulative distribution function with standard uniform marginals. Every joint distribution F of random variables X_1 and X_2 with marginal distributions U_1 and U_2 can be described by $F(x_1, x_2) = C(U_1, U_2)$, with an adequate copula $C(u_1, u_2)$ [14]. As a measure of dependence, we use a well-known rank correlation, Kendall's $\tau \in [-1,1]$. The Kendall tau is a statistic that measures the association and rank correlation between two variables X_1 and X_2. Since this measure of concordance is invariant to any strictly increasing transformation, it can be used to measure the nonlinear dependence that cannot be measured by Pearson's linear correlation coefficient. It is possible to express the Kendall tau in term of copula C which joins the variables X_1 and X_2 as mentioned in Equation (4):

$$\tau_C = \tau(X_1, X_2) = 4\iint_{I^2} C(u_1,u_2)\,dC(u_1,u_2) - 1 = 4E\big(C(U_1,U_2)\big) - 1 \qquad (4)$$

In this work, four copulas frequently used in [2] are considered: one copula from the elliptical family, the Gaussian (Normal) copula, and three Archimedean copulas, the Gumbel copula, the Clayton copula and the Frank copula. In addition, the Beta distribution is studied due to its full flexibility in skewness of the demand distribution by using various parameters α and β. The set of possible values for the standard beta family is [0,1] and the density function is given by the Equation (5):

$$f(x)=\begin{cases}\dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} & if \quad x\in[0,1] \qquad \alpha>0 \;\; et \;\; \beta>0\\ 0 & otherwise\end{cases} \tag{5}$$

with $B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$

It can be easily shown that the density function is also written as presented in Equation (6):

$$\begin{aligned} f(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}1_{[0,1]} \\ &= \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}1_{[0,1]} \end{aligned} \tag{6}$$

where Γ is the Gamma function et 1 is the characteristic function of [0,1].

if $\alpha=\beta$, this distribution is symmetric (null skewness)

if $\alpha<\beta$, this distribution is asymmetric right-tailed (with positive skewness)

if $\alpha>\beta$, this distribution is asymmetric left-tailed (with negative skewness)

Admitting a great variety of forms, it allows modeling many finite support distributions. It is used for example in the PERT method. The Beta distribution ensures that the optimal total inventory level is always between 0 and 2.

We have focused then on three cases that is:

$B(4, 12)$: the case of positively skewed beta demands

$B(8, 8)$: the case of symmetric beta demands

$B(12, 4)$: the case of negatively skewed beta demands.

This choice makes it possible to create six possible combinations for the two marginal demand distributions and thus allows having marginal distributions which are non-identical. Several previous studies assume a normal distribution which cannot model skewed cases. The knowledge of marginal distributions is sufficient to determine the optimal dedicated inventory level. However, the optimal pooled inventory level depends also on the dependence structure. To better understand the effect of dependence structure, it is useful to distinguish between the dependence structure and the level of dependence, since for the same value of Kendall's can meet different dependence structures. For this reason, the Kendall's is fixed to three different values (0.2, 0.5 and 0.8) and also to four different values (0.2, 0.4, 0.6 and 0.8). Then the corresponding copula parameters are computed.

3.2. The impact of marginal demand distribution

Sensitivity analyses of dependence structure at choice of marginal distributions are performed in figure 1. A combination is achieved between the normal copula on the one hand (Kendall's = 0.2, 0.5 and 0.8) and each of three different marginal distributions (beta (4,12), beta (8,8) and beta (12,4)). It is noticed that, to a same copula, there is a difference between different graphs indicating that the inventory level and the sign of pooling effect are influenced not only by the dependence structure but also by the marginal demand distributions.

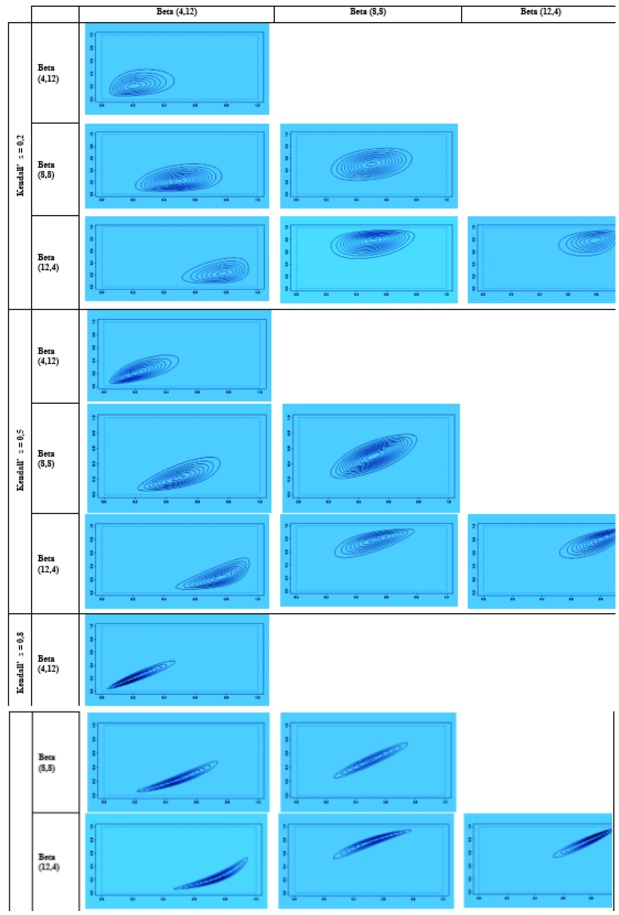

Fig. 1. Sensitivity of the dependence structure to the choice of marginal distributions: Combination of the normal copula (Kendall τ = 0.2, 0.5 and 0.8) with three different marginal distributions (beta (4, 12), beta (8, 8) and beta (12, 4)).

Comparing the graphs of different copulas under the same Kendall's (see Fig 2), strong differences in dependence structure is found. As Kendall tau increases, the densities tend to concentrate around the 45-degree line. Gumbel and Joe copulas are appropriate to model bivariate data in which it is slightly more likely that high-level demands are correlated in the upper tails. Clayton copula model's bivariate data where the dependence in the lower tails is important, perhaps due to unfavorable market conditions that affect all demand sources (i.e., low-level demands are more correlated). On the other hand, Frank copula presents a more dispersed structure and model cases where dependence is similar in high and low level demands (i.e., it is symmetric at both tails).

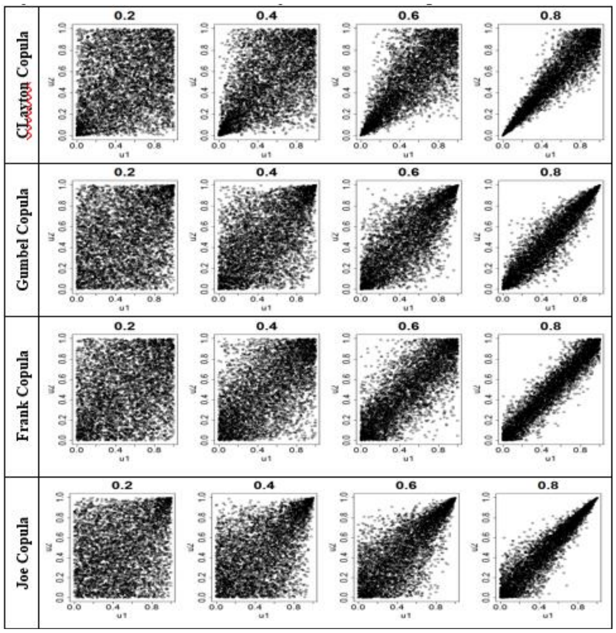

Fig. 2. Examples of different dependence structures between two random variables following Archimedean copula (Clayton, Gumbel, Frank or Joe) and for different Kendall's values.

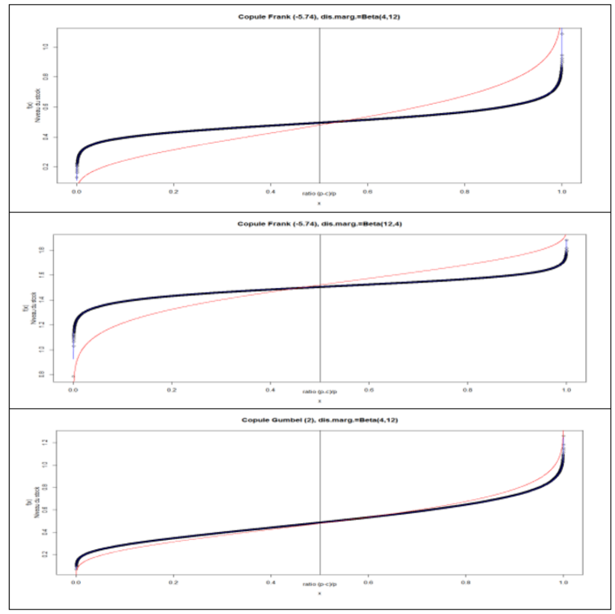

Fig. 3. Total inventory level before (red line) and after (blue line) pooling under margin ratio ((p-c)/p) and under marginal distributions and copula used.

Fig. 3 illustrates how both the pooled inventory level (curve 1 in blue) and the sum of dedicated inventory levels (curve 2 in red) change under margin ratio. This figure gives the intuition on how the dedicated and also pooled inventory levels change when margin ratio changes, i.e., when risk taking is more or less costly. We can observe that the total dedicated inventory level increases more rapidly with increase of the margin ratio than the pooled inventory level.

The pooled inventory level is more robust when values of margin ratio are medium and more sensitive when values of margin ratio are either too small (converge to zero) or too high (converge to one). If margin ratio converges to zero, then inventory level approximate more quickly to zero. On the other side, if margin ratio converges to one, then inventory level approximate more quickly to the value two.

The gap between the points of curve 1 and those of curve 2 represents the effect of pooling for a given ratio. If a point of curve 1 is above a point of curve 2 for the same ratio, then a positive gap and therefore a positive pooling effect can be observed. A point in curve 1 is below a point on curve 2 for the same ratio, which indicates a negative pooling effect. The intersection of the curves means a null pooling effect. In this case, there is no difference between pooling or independent system since the two strategies give the same inventory level. We can observe that the pooling effect is always negative if and only if the margin ratio is greater than the threshold t_0 (Proposition 1). From Fig. 3 we see that, even if the marginal demands are not Normal distributed, then pooling leads to lower inventory if and only if the margin ratio is higher than t_0 (Proposition 2).

From Fig. 3, we can observe that the shape of the two curves, the threshold as well as the magnitude of pooling depends not only on the margin ratio but also on the marginal distribution and the copula function used to model the vector demands.

The threshold value and the amplitude of pooling effect are affected by the skewness of demand distribution. The threshold t_0 can either be greater or less than the reference value 0.50 depending on the variation of the asymmetric coefficient. In the general case, a negative skewness coefficient indicates a threshold which tends to increase. On the other hand, a positive skewness coefficient results in a lower threshold. Indeed, the consolidation will result in a higher inventory level for lower levels of the marginal ratio when a demand source is more likely to be weaker than higher. This result can be explained by the fact that the right skewed distributions having a mass concentrated on the lower left part of the joint distribution while the left skewed distributions focus on the uppermost right part. For two identical left-skewed demand distributions, the consolidation effect is positive for low values of the threshold t. This result is coherent with the work of Yang and Schrage [19] who found the same result with low values of threshold t but for two demands identically and independently distributed and shifted to the left. Aydin et al. extend this finding to non-identical marginal demands [2]. They find that:

The change of one of the marginals from left skewed to right skewed while keeping the other one fixed, the threshold decreases, implying a positive pooling effect over a larger set of margin ratio.

- The left skewness of marginals increases the threshold.
- The case with one demand marginal being left-skewed and the other being right-skewed leads to similar results to the case wherein both marginals are symmetric around the mean.

The magnitude of consolidation effect decreases as any or both of the demand distributions change from being positively skewed to negatively skewed. However, pooling changes mainly the inventory levels when both of the marginal distributions are positively skewed and also when margin ratio is low.

3.3. The impact of dependence structure

3.3.3. Gumbel copula

For the high dependence case (kendall's is equal to 0.80), pooling does not have a strong effect on inventory levels for most margin ratios and for all combination of marginal distributions (the yellow curve in the Fig.4). One exception (where this curve moves upwards with respect to the x-axis) is when the margin ratio is very low therefore approaches 0 (positive pooling effect). For any marginal density combination, low (high) margin ratios result in positive (negative) pooling effect. We observe that as the dependence increases, the threshold value decreases, implying a positive pooling effect for even smaller values of margin ratio.

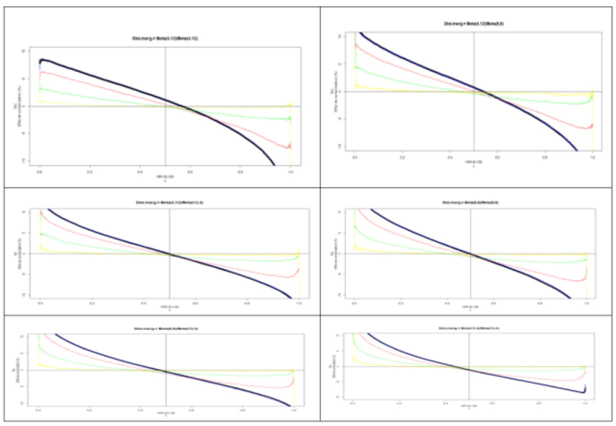

Fig. 4. Percentage pooling effect on inventory levels under Gumbel copula corresponding to a Kendall tau is equal to 0 (blue), 0.2 (Red), 0.5 (green) or 0.8 (yellow).

3.3.4. Clayton copula

Compared to Gumbel copula, two important differences are found (see Fig.5):

- Clayton copula implies a higher threshold value t_0 compared to Gumbel, given the same marginal distributions.
- The threshold value increases as the level of dependence increases. This observation is the opposite with the Gumbel copula.

The Clayton copula models the dependence between the lowest extreme values of the demands while the Gumbel copula focuses on the dependence between the highest extreme values, which is the underlying reason behind these differences.

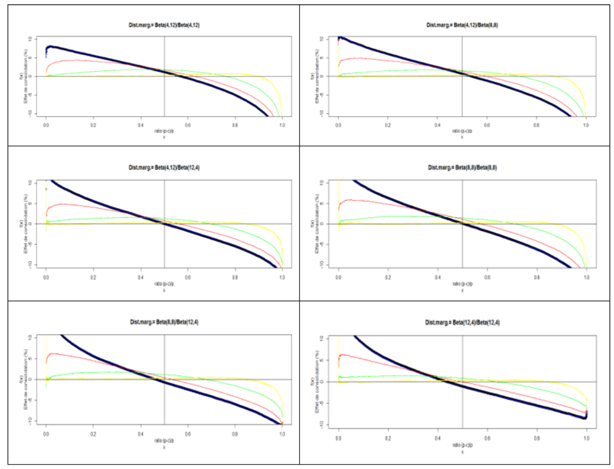

Fig. 5. Percentage pooling effect on inventory levels under Clayton copula corresponding to a Kendall tau is equal to 0 (blue), 0.2 (Red), 0.5 (green) or 0.8 (yellow).

3.3.5. Frank copula

This copula is able to cover both positive and negative dependence cases. Compared to the first two copulas (Clayton and Gumbel), the threshold value t_0 is more robust to the asymmetric of the demand distributions. This is because the symmetric of Frank copula's tail dependencies. Indeed, the pooling effect lines can show the effect of this symmetry at different levels of t.

The threshold value is not unique as Kendall's is equal to 0,8. The pooling effect for six demand distribution combinations is illustrated in figure 6. The top of Fig. 6 depicts the regions of the pooling effect where it is positive or negative. Pooling needs higher inventory levels in two different disjoint regions of the margin ratio for all combination. Therefore, when there is very high dependence, the uniqueness of the threshold value is not valid for this particular copula. In this case, the amplitude of pooling is quite weak because the co-monotonicity leads to no pooling effect.

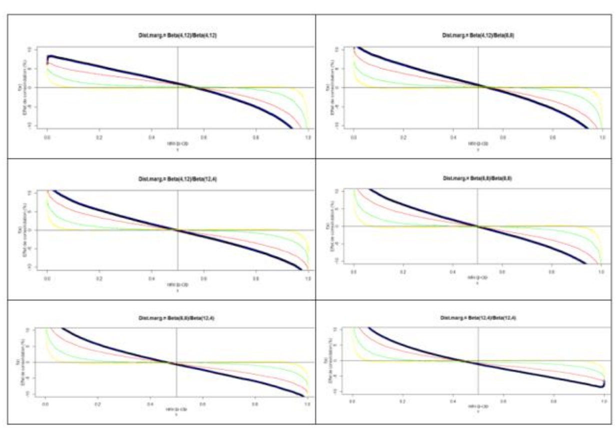

Fig. 6. Percentage pooling effect on inventory levels under Frank copula corresponding to a Kendall tau is equal to 0 (blue), 0.2 (Red), 0.5 (green) or 0.8 (yellow).

3.3.6. Joe copula

For the high-dependence case (kendall's is equal to 0.80), pooling does not have a strong effect on inventory levels for most margin ratios (the yellow curves in the Fig.7). One exception is when the margin ratio is quite low (where this curve moves upwards with respect to the x-axis): for any marginal density combination, low margin ratios result in positive pooling effect. Another observation is that as the dependence decreases, the threshold value increases, implying a positive pooling effect for even smaller values of margin ratio when dependence is high.

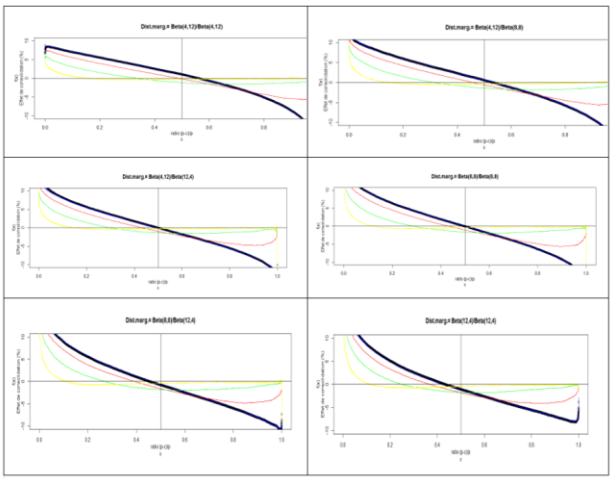

Fig. 7. Percentage pooling effect on inventory levels under Joe copula corresponding to a Kendall tau is equal to 0 (blue), 0.2 (Red), 0.5 (green) or 0.8 (yellow).

3.4. The impact of Kendall's tau variation

To discuss the sensitivity of the results with respect to Kendall tau, Table 1 illustrates the total inventory level for four copulas and symmetric demands (beta (8, 8)). It shows whether the optimal pooled inventory level varies not only with the degree of dependence but also with the dependence structure and the margin ratio. Indeed, the copula as well as its parameter (Kendall's tau) have an impact on the determination of inventory level. This table clearly shows the importance of margin ratio:

- For a low margin ratio, we see that all for copulas we tried show higher inventory levels compared to dedicated.
- For high margins, we see that all for copulas we tried show lower inventory levels compared to dedicated.
- For medium margin ratios, we see the slight difference between Frank copula and the case of independence. In the case of Clayton copula, the inventory level is higher than in the independence case especially for Kendall rates close to 50%. On the other hand, for Gumbel and Joe copulas, the required level is lower than in the independence case especially for Kendall rates close to 50%. The inventory level in dependence case will be close to its level in independence case for Kendall tau values close to 0 or 1.

- **Summary of results and managerial insights**

- In this section, we answer our research questions and summarize related managerial insights.
- The pooling effect is always negative if and only if the margin ratio is greater than the threshold t_0 (Proposition 1).
- Even if the marginal demands are Beta distributed, then pooling leads to lower inventory (negative pooling effect) if and only if the margin ratio is higher than t_0 (Proposition 2).
- The sensitivity analyses of dependence structure are changed at different marginal distributions (beta (4,12), beta (8,8) and beta (12,4)). This is indicating that the inventory level and the sign of pooling effect are influenced not only by the dependence structure but also by the marginal demand distributions (Fig. 1).
- The pooled inventory and dedicated inventory levels, the threshold as well as the magnitude of pooling depends not only on the margin ratio but also on the marginal distribution and the copula function used to model the vector demands (Figs. 2 and 3).
- The amplitude of pooling decreases when any or both demand distributions change from being positively skewed to negatively skewed. We observe also that the magnitude of pooling decrease when both of demand distributions are negatively skewed and mainly when margin ratio is small (Fig. 3).
- From Gumbel and Joe copulas: We prove that the threshold value increases as the level of dependence decreases. For all marginal distributions combination, we prove that, for lowest margin ratio values, the pooling effect is positive and more the demand dependence decreases (increases) more the effect will be great (small). Therefore, pooling leads to higher inventory levels than the no pooling system.
- Further, for all marginal distributions combination, we prove that, for highest margin ratio values, the pooling effect is negative and more the demand dependence decreases (increases) more the effect will be great (small). In this case, pooling leads to lower inventory levels than the no pooling system. In consequence, the inventory levels are more likely to become pooling when the margin ratio is high and this for high or low demand dependence, or when the margin ratio is low but under high demand dependence (Figs. 4 and 7).
- From Clayton copula: We conclude a higher threshold value t_0 compared to Gumbel copula, given the same marginal distributions. We prove that the threshold value increases as the level of dependence increases. On the other side, we prove the same results as the Gumbel copula that the inventory levels are more likely to become pooling when the margin ratio is high and this for high or low demand dependence, or when the margin ratio is low but under high demand dependence (Fig. 5).
- From Frank copula, the threshold value t_0 is more robust to the skewness of the marginals as compared to the three copulas (Gumbel, Clayton and Joe). The pooling effect requires higher inventory levels in two different disjoint regions of the margin ratio. In consequence, under certain conditions, this copula is not well performed (Fig. 6).
- The optimal pooled inventory level varies not only with the degree of dependence but also with the dependence structure (copula function) and the margin ratio:

- For a low margin ratio, the four copulas (dependence case) give a higher inventory compared to no pooling case.
- For a high margin ratio, the four copulas give a lower inventory compared to the independent system.
- For medium margin ratios: The Frank copula case approach the no pooling case for all values of kendall's tau. As τ approaches 0 or 1, the inventory level in dependence case (the four copulas) will be close to its level in independence case. As τ approaches 0.5, the Clayton copula case leads to higher inventory levels than the independence case, while the Gumbel and Joe copulas case lead to lower inventory levels than the decentralized case. Further, we prove that the inventory level increase with increasing of the margin ratio and this for all values of kendall's tau (Table1).

- Consequently, under certain conditions, the Gumbel and Joe copulas performs better in terms of the inventory pooling than the Clayton copula. Further, the choice of the Frank copula is worse in terms of inventory pooling.

Table 1. The total inventory level following Kendall tau for four copulas and three margin ratios assuming symmetric demands (beta (8, 8)).

Marg. ratio	Kendall tau / Copula	0,01	0 ,10	0,20	0,30	0,40	0,50	0,60	0,70	0,80	0.90	0.99
0.20	***Indep.***	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134	0.7887134
	Clayton	0.8521518	0.8443393	0.8365289	0.8251038	0.8133046	0.8035212	0.7988664	0.794213	0.7914873	0.7888778	0.811803
	Gumbel	0.8518814	0.8424653	0.8327185	0.8238637	0.8132672	0.8055041	0.8009153	0.7974297	0.7900879	0.7892973	0.7885455
	Frank	0.8532654	0.8411464	0.8271979	0.8163401	0.8028684	0.7964208	0.7890276	0.7894006	0.7865778	0.7881173	1.375128
	Joe	0.8525631	0.8434877	0.8331423	0.8231211	0.8121946	0.8019102	0.7937137	0.7868338	0.7836242	0.7844102	0.7890089
0.50	***Indep.***	1	1	1	1	1	1	1	1	1	1	1
	Clayton	0.9997766	1.005339	1.008701	1.013818	1.015053	1.015522	1.012285	1.00866	1.003262	1.000175	1.019209
	Gumbel	0.9984798	0.996764	0.9946334	0.9931202	0.9940965	0.9933026	0.9953796	0.9959919	0.9988971	1.000904	0.9990992
	Frank	1.000353	1.000603	1.000418	1.000387	0.999619	1.001336	0.9995996	0.9982659	1.000357	0.9991575	0.9998197
	Joe	0.9986729	0.993921	0.9882806	0.9834164	0.981741	0.9828313	0.9862548	0.9907846	0.9976757	0.9976757	0.9987185
0.80	***Indep.***	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287	1.211287
	Clayton	1.147295	1.15747	1.167975	1.178641	1.189371	1.196682	1.205591	1.212265	1.218605	1.212974	1.223848
	Gumbel	1.148169	1.152701	1.163135	1.172218	1.181136	1.194308	1.19944	1.201607	1.209896	1.212427	1.212167
	Frank	1.147174	1.159237	1.173147	1.183853	1.195296	1.201688	1.209831	1.211959	1.213978	1.212972	1.586494
	Joe	1.146454	1.152422	1.162132	1.172948	1.18565	1.195038	1.202508	1.207981	1.209057	1.211205	1.213916

4. Conclusion

In this paper, a newsvendor inventory pooling model composed of two identical products is considered including different conditions on dependent demand. Stock levels depend on univariate distributions used for modeling marginal demands. It is then useful to model the demands via the most adequate distribution and the dependence via the most suitable copula for determining optimal inventory levels. The dependence structure, marginal demand distribution and Kendall tau have a significant impact on the sign of pooling effect and the conditions under which the pooling is preferable to a no pooling.

The results suggest, for a low margin ratio, higher inventory compared to no pooling case and this for the four copulas (dependence case). For a high margin ratio, the four copulas give a lower inventory compared to the independent system. However, for medium margin ratios, three decisions can be occurred. Firstly, the Frank copula case approach the no pooling case for all values of kendall's tau. Secondly, when the Kendall tau approaches 0 or 1, the inventory level in dependence case (the four copulas) will be close to its level in independence case. Thirdly, when the Kendall tau approaches 0.5, the Clayton copula case leads to higher inventory levels than the independence case, while the Gumbel and Joe copulas case lead to lower inventory levels than the decentralized case. Consequently, under certain conditions, the Gumbel and Joe copulas performs better in terms of the inventory pooling than the Clayton copula. Further, the choice of the Frank copula is worse in terms of inventory pooling.

Decision makers can refer to this type of analysis to choose or not choose pooling strategy based on the degree of dependence, dependence structure and degree of asymmetry of marginal demand distributions.

Finally, we highlight potential extensions for future research. In this paper, we study our model with two demands. The demand sources are greater than two (more than two retailers) in practice. A more advanced version of the copula called CDvine copula is able to model a vector of any number of demands. Hence, it is possible to extend our results to case with arbitrary number of demands and prove that the benefit of pooling most clear and most significant when number of demands be greater than two.

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